This follows by the mean value theorem for integrals and the continuity of $\bar{\lambda}$, $\dot{\bar{\lambda}}$, and $\bar{X}$ over impulses.[3] The F/m integrals are merely the velocity increments at the times t_i.

III. Question of Independence

The six constants $\bar{L}$ and $\bar{M}$ do not supply a linearly independent set of integrals because the 6×6 matrix

$$T = \begin{bmatrix} \partial\bar{L}/\partial\bar{X} & \partial\bar{L}/\partial\dot{\bar{X}} \\ \partial\bar{M}/\partial\bar{X} & \partial\bar{M}/\partial\dot{\bar{X}} \end{bmatrix}$$

has rank 5. This may be shown directly by using elementary row operations on T. The integrals $\bar{L}^*$ and $\bar{M}^*$ as defined for unpowered flight by Eqs. (4) and (9) are linear in the adjoint variables $\bar{\lambda}$ and $\dot{\bar{\lambda}}$ and therefore may be expressed

$$B\begin{bmatrix} \bar{\lambda} \\ \dot{\bar{\lambda}} \end{bmatrix} = \begin{bmatrix} \bar{L}^* \\ \bar{M}^* \end{bmatrix} \tag{12}$$

Upon calculating the matrix T one readily observes that $T = B$ from which it follows that $\bar{L}^*$ and $\bar{M}^*$ represent only five independent integrals. However, the relation given by Pines[1]

$$1/3\ \bar{\lambda}\cdot\dot{\bar{X}} + 2/3\ \dot{\bar{\lambda}}\cdot\bar{X} = b - H(t - t_o)$$

where H is the variational Hamiltonian ($F = 0$)

$$H = -(\mu/R^3)\bar{\lambda}\cdot\bar{X} - \dot{\bar{\lambda}}\cdot\dot{\bar{X}}$$

and b is an integration constant, may be adjoined to Eq. (12) to give a new linear system

$$\begin{bmatrix} B \\ 1/3\ \dot{\bar{X}}^T\ 2/3\ \bar{X}^T \end{bmatrix}\begin{bmatrix} \bar{\lambda} \\ \dot{\bar{\lambda}} \end{bmatrix} = \begin{bmatrix} \bar{L}^* \\ \bar{M}^* \\ b - H(t - t_o) \end{bmatrix} \tag{13}$$

which does have maximum rank for nonparabolic orbits. More specifically, if the 7×6 coefficient matrix of Eq. (13) is defined to be A, then it may be shown that the determinant of A^TA is

$$\mu^4R^2V^6[V^2 - 2(\mu/R)]^2$$

IV. Some Observations

For the case $F = 0$ the equations of variation of Eq. (1) defining the state transition matrices

$$\frac{\partial\bar{X}}{\partial\bar{X}_o}, \frac{\partial\bar{X}}{\partial\dot{\bar{X}}_o}, \frac{\partial\dot{\bar{X}}}{\partial\bar{X}_o}, \text{ and } \frac{\partial\dot{\bar{X}}}{\partial\dot{\bar{X}}_o}$$

satisfy Eq. (2). Thus the integrals $\bar{L}^*$ and $\bar{M}^*$ serve also as integrals for the state transition matrices. A completely analogous derivation gives the state transition analog of Eq. (13)

$$A\begin{bmatrix} \partial\bar{X}/\partial\bar{X}_o & \partial\bar{X}/\partial\dot{\bar{X}}_o \\ \partial\dot{\bar{X}}/\partial\bar{X}_o & \partial\dot{\bar{X}}/\partial\dot{\bar{X}}_o \end{bmatrix} = A_o$$

It is evident from Eq. (13) that if $H = 0$, then the adjoint variables are explicitly dependent upon the constants $\bar{L}^*$, $\bar{M}^*$, and b and the state $\bar{X}$ and $\dot{\bar{X}}$ only. Hence if $\bar{X}$ and $\dot{\bar{X}}$ are periodic and $H = 0$ then also are $\bar{\lambda}$ and $\dot{\bar{\lambda}}$ with the same period. Moreover if $H \neq 0$ and $\bar{X}$ and $\dot{\bar{X}}$ are periodic then $\bar{\lambda}$ and $\dot{\bar{\lambda}}$ are not periodic.

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Spin Required to Limit Projectile Oscillations in a Finite Range

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1. Introduction

The problem of spin stabilization is one of long standing. It was considered by Fowler (1920)[2] and the fundamental condition for complete stabilization was derived. The general equations of motion are very complex and it is difficult to extract from them simple criteria for the amount of spin necessary to restrict the yaw angle to some small amount. In a hyperballistic range, such as the one at RARDE, however the projectiles travel at very high speed over quite short distances and often at reduced pressures. In this case it is not unreasonable to simplify the problem by neglecting nonlinear effects, gravity, damping, and Magnus force. For instance the change in velocity due to gravity is about 10^{-6} projectile velocity and the changes due to damping and Magnus forces do not amount to more than a few per cent.

The simplified set of equations allow a very simple solution which shows that in this case the classical stability criterion is not sufficient to ensure that the oscillations are small. New criteria are therefore sought which will ensure this.

2. Theory

Consider an axisymmetric projectile whose moment of inertia about the axis of symmetry is I_1 and whose moment of inertia about an axis through the center of gravity normal to the axis of symmetry is I_2. We will denote the angle of yaw by α and the angle of pitch by β. If we let M be the aerodynamic pitching (or yawing as the projectile is axisymmetrical) moment per radian and let p_0 be the spin velocity then it is shown in Arnold and Maunder (1961)[1] that the perturbed motion obeys the equations

$$(I_2\ddot{\alpha} - M\alpha) - I_1p_0\dot{\beta} = 0 \tag{1a}$$

$$I_1p_0\dot{\alpha} + (I_2\ddot{\beta} - M\beta) = 0 \tag{1b}$$

when nonlinear effects, damping, gravity, and Magnus forces are neglected. Initially it will be assumed that the yaw angle is zero although the rate of yawing may be finite due to the flexing of the barrel as the projectile is emerging.

It is convenient to nondimensionalize Eq. (1) by setting

$$\tau = Vt/d,\ P = Md^2/I_2V^2, \text{ and } Q = I_1\,p_0d/I_2V \tag{2}$$

where V is the projectile velocity and d is the maximum projectile diameter.

The initial conditions can then be written as

$$\alpha = \beta = \beta' = 0 \text{ and } \alpha' = \Omega \text{ when } \tau = 0 \tag{3}$$

(For brevity Ω will be referred to as gun jump). The solution satisfying these conditions can easily be shown to be

$$\alpha = (2\Omega/\lambda)\cos(Q\tau/2)\sin(\lambda\tau/2) \tag{4a}$$

$$\beta = -(2\Omega/\lambda)\sin(Q\tau/2)\sin(\lambda\tau/2) \tag{4b}$$

where

$$\lambda = (Q^2 - 4P)^{1/2}$$

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The classical stability criterion can be written in the form

$$Q^2 > 4P \tag{5}$$

This ensures that λ is real and thus there are no exponentially growing modes. The amplitude of the yawing oscillation may nevertheless be very large if λ is small. A criterion of more practical use is to restrict the yawing amplitude to less than ϵ rad. The condition for this is then

$$\lambda > 2\Omega/\epsilon \tag{6}$$

or

$$Q^2 > 4[P + (\Omega/\epsilon)^2] \tag{7}$$

If the range distance is quite short it may be possible to allow the maximum oscillation to be large or even grow exponentially provided the amplitude does not increase too much in the time available. If $4P < Q^2 < 4[P + (\Omega/\epsilon)^2]$, the amplitude of the oscillation is large but the period of the sinusoidal component $\sin(\lambda\tau/2)$ also becomes very large. From Eq. (4) it follows that the yaw or pitch of the projectile will not exceed ϵ rad. within a time τ provided

$$\theta < \text{maximum value of } \epsilon\psi/\sin\psi \tag{8}$$

where

$$\theta = \Omega\tau \text{ and } \psi = \lambda\tau/2 \tag{9}$$

If the gun jump parameter is given then the oscillation can be restricted to less than ϵ rad. provided ψ is greater than some number $\delta(\theta,\epsilon)$, say

$$Q^2 > 4[P + (\delta/\tau)^2] \tag{10}$$

Note that if $\theta > \pi\epsilon/2$, Eqs. (7) and (10) coincide. If

$$4P > Q^2, \text{ let } \mu = (4P - Q^2)^{1/2} = i\lambda \tag{11}$$

Writing $-i\mu$ for λ in Eq. (4) the condition that the oscillation be restricted to less than ϵ rad. is now

$$\theta < \epsilon\phi/sh\phi \tag{12}$$

where

$$\phi = \mu\tau/2 \tag{13}$$

For a given gun jump parameter θ, which is less than ϵ, the oscillation can be restricted to less than ϵ rad. provided ϕ is less than some number γ (θ,ϵ) say [if θ is very small $\gamma(\theta)$ is just $-\ln\theta$], or

$$Q^2 > 4[P - (\gamma/\tau)^2] \tag{14}$$

The preceding results may be summarized and classified according to the magnitude of the gun jump parameter θ.

1) $\theta < \epsilon$ Exponentially growing modes may be allowed provided Q satisfies

$$Q^2 > 4\{P - [\gamma(\theta,\epsilon)/\tau]^2\} \text{ At } \theta = \epsilon \qquad \gamma = 0$$

2) $\theta > \epsilon$ Only stable oscillations can be allowed and Q must also satisfy

$$Q^2 > 4\{P + [\delta(\theta,\epsilon)/\tau]^2\} \text{ At } \theta = \epsilon \qquad \delta = 0$$

$$\text{for } \theta > \pi\epsilon/2, \ \delta = \theta/\epsilon$$

3. Application to the Hyperballistic Range

The velocity V of a projectile is constant if we neglect gravity and drag forces and thus the time of flight down a range of length R is simply (R/V). Substituting this value in Eq. (2) shows that the nondimensional time of flight is

$$\tau = R/d \tag{15}$$

The pitching moment can be written

$$M = \tfrac{1}{2}C_{M\alpha}\rho_a V^2 d^3 \tag{16}$$

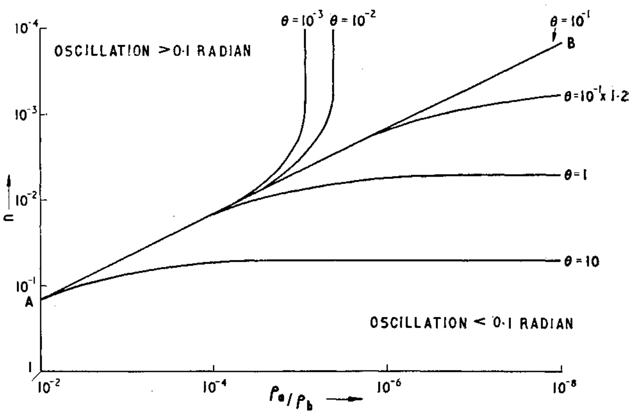

Fig. 1 Relation between spin, gun jump, and air density for the yawing amplitude to be less than 0.1 rad; $c = 3$, $\tau = 4000$.

where

$C_{M\alpha}$ = pitching moment coefficient

ρ_a = air density

It is also convenient to set

$$I_1 = k_1\rho_b d^5 \text{ and } I_2 = k_2\rho_b d^5 \tag{17}$$

where

k_1,k_2 = dimensionless constants depending on the projectile geometry

ρ_b = projectile density

Thus from Eq. (2) we find

$$P = \frac{\rho_a}{\rho_b}\frac{C_M}{2k_2} \text{ and } Q = \frac{k_1}{k_2}\frac{p_0 d}{V} \tag{18}$$

If the projectiles are long cylinders, let the ratio of length to diameter be c then for this case

$$k_1 = (\pi/32)c, \ k_2 = (\pi/48)c^3 \tag{19}$$

Also let the number of turns the projectile makes per diameter length of flight be n then

$$p_0 d/V = 2\pi n \tag{20}$$

and

$$Q = 3\pi n c^{-2} \tag{21}$$

Substituting Eqs. (19) into Eq. (18) and taking C_M approximately equal to 2 gives

$$P = (48/\pi c^3)(\rho_a/\rho_b) \tag{22}$$

For a typical hyperballistic range $\tau \approx 4000$ and values of c usually range from about 3 to 10.

Figures 1 and 2 show the relationship between spin, gun jump, and air density for the oscillation amplitude to be less than 0.1 rad. in the two cases when $c = 3$ and $c = 10$, respectively and $\tau = 4000$. For a given θ the oscillation will be restricted to less than 0.1 rad. provided the values of n and ρ_a/ρ_b correspond to a point below and to the right of the curve for this θ.

Let us consider first the growing modes. For this region it may be noted that if the gun jump parameter $\theta(= R\dot{\alpha}/V)$ is greater than 10^{-1} the yawing amplitude cannot be contained within 0.1 rad. for a range length $R/d = \tau = 4000$. When $\theta = 10^{-1}$ the spin-density relationship is just the line AB whereas for smaller values of θ the relationship follows

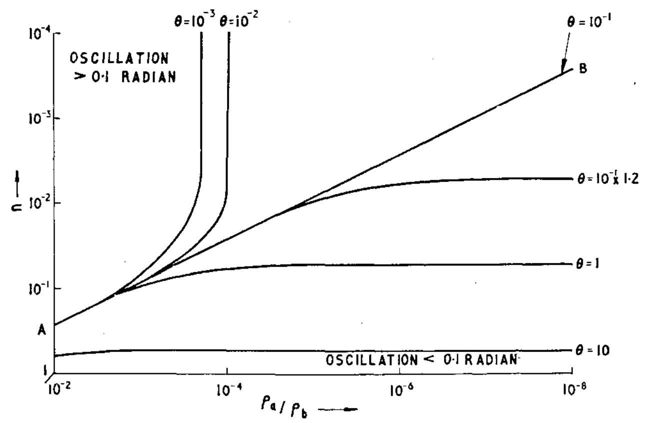

Fig. 2 Relation between spin, gun jump, and air density for the yawing amplitude to be less than 0.1 rad; $c = 10$, $\tau = 4000$.

the line AB and then breaks away to asymptote at a critical density. With such low values of θ and for densities lower than the critical no spin is required to restrict the oscillation of 0.1 rad. The critical density ratio for a given gun jump θ is given by

$$(\rho_a/\rho_b) = (\pi/48)[c^3\gamma(\theta)^2/\tau^2] \tag{23}$$

If the gun jump parameter θ is greater than 10^{-1} then in order to restrict the yawing to less than 0.1 rad. n must be increased until the spin density point lies in the region of stable oscillations and below the line for the gun jump parameter. At sufficiently low densities this line is independent of density. This density is given approximately by

$$(\rho_a/\rho_b) = (\pi/48)[c^3\delta(\theta)^2/\tau^2] \tag{24}$$

and the minimum spin is then

$$n = (2/3\pi)(c^2\delta/\tau) \tag{25}$$

Conclusion

In a hyperballistic range where the projectile velocity is high, the time of flight short and the range density often quite low the equations of motion of a spinning projectile may take a very simple form whose solution is given in Eq. (4). Examining this solution shows that the classical stability criterion is not sufficient to ensure that the yawing oscillations are small. Revised criteria which restrict the yawing oscillation to ϵ radian are given in Eqs. (10) and (14).

Detailed calculations are presented in Figs. 1 and 2 for a range length $\tau(= R/d)$ of 4000 and projectile (length/diam) ratios of 3 and 10. It has been shown in these cases that yawing can theoretically be restricted to less than 0.1 radian provided the following conditions hold.

1) If the gun jump parameter $\theta(= R\dot{\alpha}/V)$ is less than 0.1 then the minimum spin at a given density must correspond to a point on the curves of appropriate θ above the line AB. There exists a critical density, given by Eq. (23), below which it is unnecessary to spin the projectile at all for that amount of gun jump.

2) If the gun jump parameter θ is greater than 0.1 then the minimum spin at a given density must correspond to a point on the curve of appropriate θ below the line AB. At densities below the value given in Eq. (24) the minimum spin is constant at the value given in Eq. (25).

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Twisting of a Composite Spherically Aeolotropic Sphere with an Isotropic Core

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Introduction

In this paper the problem of twisting of a sphere of a spherically aeolotropic material is considered when there is an isotropic core. In the shell the planes of symmetry at each point are taken to be perpendicular to the directions of spherical polar coordinates (r,θ,ϕ). The shell is bounded by surfaces $r = a$ and $r = b$. The composite body is twisted by tangential forces on the surface $r = b$, the traction on the hemispherical surface $0 \leqslant \theta \leqslant \pi/2$ being equivalent to a couple about the polar axis from which θ is measured. Stresses are found in the aeolotropic shell as also in the isotropic core. Particular cases are deduced 1) when there is a cavity inside and 2) when the inclusion is rigid. The first case was given by Chakravorty[1] without any numerical computation. Extensive numerical results are given showing the concentration of stress on the boundary $r = a$ of the inclusion when the outer material is topaz (spherically aeolotropic) and the inner material is steel (isotropic). It is seen that the stress concentration is maximum at the equatorial region. Effects of cavity or rigid inclusion, on stresses, are shown in tabular form.

Problem and Its Solution

We use spherical polar coordinates r,θ,ϕ, the centre of the shell being taken as the pole. The strain energy function of such an aeolotropic material is given by

$$2W = c_{11}e_{rr}^2 + c_{22}e_{\theta\theta}^2 + c_{33}e_{\phi\phi}^2 + 2c_{23}e_{\theta\theta}e_{\phi\phi} + 2c_{13}e_{\phi\phi}e_{rr} + 2c_{12}e_{rr}e_{\theta\theta} + c_{44}e_{\theta\phi}^2 + c_{55}e_{\phi r}^2 + c_{66}e_{r\theta}^2 \tag{1}$$

where c_{ij}'s are elastic coefficients.

For twisting of the body by couples about the polar axis from which θ is measured, we assume the displacement components as

$$u = v = 0, w = R \sin 2\theta \tag{2}$$

where R is a function of r only. For the outer shell which is spherically aeolotropic the stresses are

$$\begin{aligned} \sigma_r = \sigma_\theta = \sigma_\phi = \tau_{r\theta} &= 0 \\ \tau_{\theta\phi} &= -2c_{44}(R/r)\sin^2\theta \\ \tau_{\phi r} &= c_{55}(dR/dr - R/r)\sin 2\theta \end{aligned} \tag{3}$$

Table 1 Values of $-\pi a^3\tau_{\theta\phi i}/M$ for certain values of θ and δ in the composite case

	δ				
θ	0.2	0.4	0.6	0.8	1.0
15°	0.0014	0.0029	0.0043	0.0058	0.0072
30°	0.0054	0.0108	0.0161	0.0215	0.0269
45°	0.0108	0.0215	0.0323	0.0430	0.0538
60°	0.0161	0.0323	0.0484	0.0646	0.0807
75°	0.0201	0.0402	0.0602	0.0803	0.1004
90°	0.0213	0.0426	0.0640	0.0853	0.1067

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